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Multicriteria Planar Ordered Median Problems[1]

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Abstract. In this paper, we deal with the determination of the entire set of Pareto solutions of location problems involving Q general criteria. These criteria include median, center, or centdian objective functions as particular instances. We characterize the set of Pareto solutions of all these multicriteria problems for any polyhedral gauge. An efficient algorithm is developed for the planar case and its complexity is established. Extensions to the nonconvex case are also considered. The proposed approach is more general than previously published approaches to multicriteria location problems.

Key Words. Location theory, multicriteria optimization, algebraic optimization, geometrical algorithms.

1. Introduction

In the process of locating a new facility, usually more than one decision maker is involved. This is due to the fact that typically the cost connected to the decision is relatively high. Of course, different persons may have different (conflicting) objectives. On other occasions, different scenarios must be compared in order to be implemented or simply the uncertainty in the parameters leads to the consideration of different replications of the objective function. If only one objective has to be taken into account, a broad range of models is available in the literature (see Ref. 1). In contrast to that, only a few papers have looked at more realistic models for facility location, where multiple objectives are involved (see Refs. 2–5).

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One of the main deficiencies of the existing approaches is that only a few (in most papers, one) different types of objectives can be considered and the solution approaches depend very much on the specific chosen metric. Also, a detailed complexity analysis is missing in most of the papers.

On the other hand, there is a clear need for flexible models where the complexity status is known, since these are prerequisites for the successful implementation of a decision support system for location planning which can be used by the decision makers. In this paper, we present a model for continuous multicriteria location problems which fulfills the requirement of flexibility with respect to the choice of objective functions. To this end, we present a new type of objective function (called ordered median function), developed in Refs. 6–9, which includes most of the classical location objective functions as special cases, for instance, the Weber, center, centdian, k-centrum objective functions and the Weber objective function with positive and negative weights.

Additionally, we allow the use of polyhedral gauges as distance functions in each objective function. It should be mentioned that, by the polyhedral gauge approach, we are able to approximate every gauge (see Ref. 10). Therefore, the solution set of any problem with general gauges can be approximated by the solution set of the same problem with polyhedral gauges approaching the original ones (see Ref. 11). The outline of the rest of the paper is as follows.

In Section 2, the problem is formally introduced and basic tools and definitions are presented. Section 3 is devoted to the bicriteria case in the plane; Sections 4 and 5 extend these results to three-criteria case and then to the general planar Q-criteria case. The paper ends with concluding remarks on extensions to the nonconvex case and an outlook to future research. Throughout the paper, we keep track of the complexity of the presented algorithms.

2. Basic Tools and Definitions

First, we restate some definitions which are needed throughout the paper.

Denote the set of demand points by

$$A := \{a_1, \ldots, a_M\}.$$

Let

$$B_i \subset \mathbb{R}^2, \quad \text{for } i \in \mathcal{M} := \{1, 2, \ldots, M\},$$

be a compact, convex set containing the origin in its interior. The gauge with respect to B_i is defined as

$$\gamma_i : \mathbb{R}^2 \to \mathbb{R}, \quad \gamma_i(x) := \inf\{r > 0 \ : \ x \in rB_i\}.$$

The polar set B_i^o of B_i is given by

$$B_i^o := \{p \in \mathbb{R}^2 : \langle p, x\rangle \le 1, \forall x \in B_i\},$$

the normal cone to B_i at x is given by

$$N(B_i, x) := \{p \in \mathbb{R}^2 : \langle p, y - x\rangle \le 0, \forall y \in B_i\};$$

the boundary of B_i is denoted by $\mathrm{bd}(B_i)$.

In this paper, we consider mainly the case where each γ_i, with $i \in \mathcal{M}$, is a polyhedral gauge, which means that B_i is a convex polytope with extreme points

$$\mathrm{Ext}(B_i) := \left\{e_1^i, \ldots, e_{G_i}^i\right\}.$$

The maximal number of extreme points is denoted by

$$G_{\max} := \max\{G_i : i \in \mathcal{M}\}.$$

We define the fundamental directions $d_1^i, \ldots, d_{G_i}^i$ as the halflines determined by 0 and $e_1^i, \ldots, e_{G_i}^i$. Let $\pi = (p_i)_{i \in \mathcal{M}}$ be a family of elements of $\mathbb{R}^2$ such that

$$p_i \in B_i^o, \ \text{for each}\, i \in \mathcal{M},$$

and let

$$C_\pi = \bigcap_{i \in \mathcal{M}} (a_i + N(B_i^o, p_i)).$$

According to Ref. 12, a nonempty convex set C is called an elementary convex set if there exists a family π such that $C_\pi = C$.

It should be noted that, if the unit balls are polytopes, we can obtain the elementary convex sets as the intersections of cones generated by the fundamental directions of these balls pointed at each demand point (for details, see Ref. 12). Two dimensional elementary convex sets are called cells. Therefore, each cell is a polyhedron and its vertices are the intersection points. Finally, in the case $\mathbb{R}^2$, there exists an upper bound on the number of cells, which is $O((MG_{\max})^2)$, see Ref. 12.

2.1. General Approach: Ordered Median Problem. In this section, we present a general location model, the ordered median problem, introduced in Refs. 7–8 and later elaborated for the polyhedral case in Ref. 9.

Consider the set of demand points $A=\{a_1,\dots,a_M\}$, the corresponding gauges $\gamma_i(\cdot), i\in\mathcal{M}$, and two sets of nonnegative scalars $\Omega:=\{\omega_1,\dots,\omega_M\}$ and $\Lambda:=\{\lambda_1,\dots,\lambda_M\}$; the elements $\omega_i, i\in\mathcal{M}$, are the weights of the importance given to the existing facilities $a_i, i\in\mathcal{M}$; the elements of Λ allow one to choose among different kinds of objective functions. Given a permutation σ of $\mathcal{M}$ verifying

$$\omega_{\sigma(1)}\gamma_{\sigma(1)}(x-a_{\sigma(1)})\le\omega_{\sigma(2)}\gamma_{\sigma(2)}(x-a_{\sigma(2)})\le\cdots\le\omega_{\sigma(M)}\gamma_{\sigma(M)}(x-a_{\sigma(M)}),$$

we denote

$$d_{(i)}(x):=\omega_{\sigma(i)}\gamma_{\sigma(i)}\left(x-a_{\sigma(i)}\right).$$

Notice that the order of the sequence depends on the point x.

The ordered median problem is given by the following formulation :

$$\min_{x\in\mathbb{R}^2} F(x):=\sum_{i=1}^{M}\lambda_i d_{(i)}(x). \tag{1}$$

The set of optimal solutions of this problem is called $\mathcal{X}^*(F)$ or simply $\mathcal{X}^*$ if this is possible without causing confusion. This objective function looks very much like the objective function of the classical Weber problem; but in fact, this function is pointwise defined and in general nonconvex as the following example shows.

Example 2.1. Consider two demand points $a_1=(0,0)$ and $a_2=(10,5)$, weights $\lambda_1=100$ and $\lambda_2=1$ with l_1-norm and $\omega_1=\omega_2=1$. We obtain two optimal solutions to Problem (1), lying in each demand point. Therefore, the objective function is not convex, since we have a nonconvex optimal solution set.

$$\begin{aligned}F(a_1)&=\lambda_1\gamma(a_1-a_1)+\lambda_2\gamma(a_1-a_2)\\&=100\times 0+1\times 15=15,\\F(a_2)&=\lambda_2\gamma(a_2-a_1)+\lambda_1\gamma(a_2-a_2)\\&=1\times 15+100\times 0=15,\\F((1/2)(a_1+a_2))&=100\times 7.5+1\times 7.5=757.5.\end{aligned}$$

Nevertheless, if we assume that the λ-weights are in nondecreasing sequence, we obtain that the objective function is convex; see Ref. 8 for more details. The reader may note that the ordered median problem is a general objective function because it includes as particular instances the Weber problem ($\lambda_1 = \lambda_2 = \cdots = \lambda_M = 1$), the α-centdian problem ($\lambda_1 = \cdots \lambda_{M-1} = 1-\alpha$ and $\lambda_M = 1$), and the center problem ($\lambda_1 = \cdots = \lambda_{M-1} = 0$ and $\lambda_M = 1$) among others.

Moreover, new useful objective functions can be modelled easily. For example, assume that we are not only interested in minimizing the distance to the most remote client (center objective function), but instead we would like to optimize the average distance to the k most remote clients. This can be modelled easily by setting $\lambda_{M-k-1}, \ldots, \lambda_M = 1$ and all other λ's to zero. This k-centrum problem is a different way of combining the average behavior and worst-case behavior. Also, ideas from robust statistics can be implemented by discarding the k_1 nearest and simultaneously the k_2 farthest clients ($k_1 + k_2$ trimmed mean). Moreover, the direct relation of the λ's to the choice of the objective function is quite useful for scenario analysis.

In what follows, we assume that the λ-weights are given in nondecreasing sequence. Notice that the computation of $F(x)$ is not a trivial task. We do not have an explicit formula of F as a linear function of the distances in $\mathbb{R}^2$, because we have different expressions for F depending on the order in the sequence of distances. However, F behaves as the classical Weber function in a region where this order does not change. To this end, we use the concept of bisectors and ordered regions.

The set $B(a_i, a_j)$, $i \neq j$, consisting of points

$$\left\{x \in \mathbb{R}^2 : \omega_i \gamma_i (x - a_i) = \omega_j \gamma_j (x - a_j)\right\},$$

is called the bisector of a_i and a_j with respect to (ω_i, γ_i) and (ω_j, γ_j). Note that

$$B(a_i, a_j) = B(a_j, a_i);$$

see Figure 1, where $B(a_i, a_j)$ is denoted by B_{ij}.

Given a permutation σ on the index set $\mathcal{M}$, the ordered region O_σ is given by

$$O_\sigma := \left\{x \in \mathbb{R}^2 : \omega_{\sigma(1)} \gamma_{\sigma(1)} \left(x - a_{\sigma(1)}\right) \leq \cdots \leq \omega_{\sigma(M)} \gamma_{\sigma(M)} \left(x - a_{\sigma(M)}\right)\right\}. \tag{2}$$

Note that the concept of ordered regions can be seen as an extension to classical Voronoi diagrams. The set that we obtain as intersection

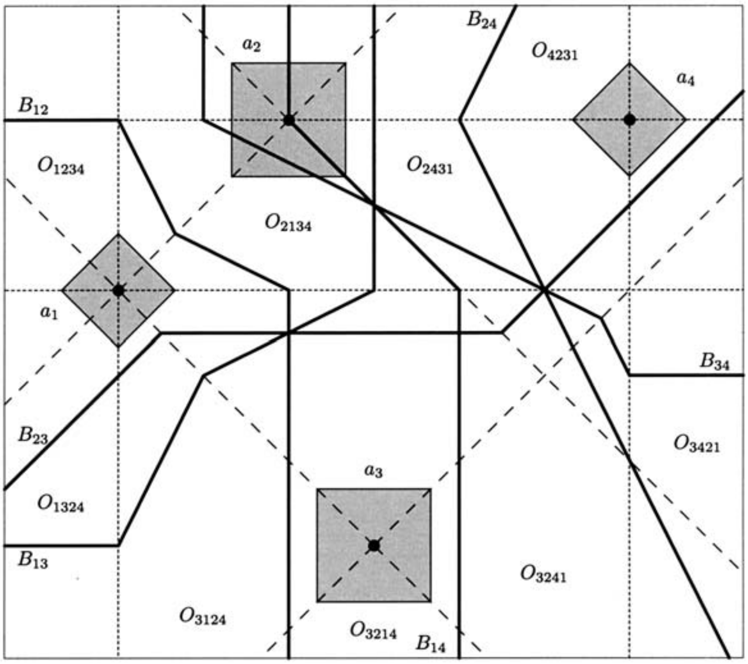

Fig. 1. Bisector lines and ordered regions generated by four existing facilities $a_1, \ldots, a_4$ associated with the l_1-norm respectively the l_∞-norm for $\Omega := \{1, 1, 1, 1\}$.

of an elementary convex set and an ordered region is called an ordered elementary convex set. The vertices of the ordered elementary convex sets are called generalized intersection points and the set that contains all of them is denoted by $\mathcal{GIP}$. Two-dimensional ordered elementary convex sets are called cells. Finally, the set of all the cells is denoted by $\mathcal{C}$.

Reference 8 obtained that the objective function of the ordered median problem is linear in each ordered elementary convex set. Therefore, there exists an optimal solution of the ordered median problem in $\mathcal{GIP}$. In the case of polyhedral gauges with at most $G_{\max}$ fundamental directions, an upper bound of the number of ordered elementary convex sets in $\mathbb{R}^2$ is $O(M^4 G_{\max}^2)$; see Ref. 9.

2.2. Multicriteria Problems and Level Sets. Let $F^1, \ldots, F^Q$ be functions from $\mathbb{R}^2$ to $\mathbb{R}$. If we want to optimize simultaneously all these objective functions, we get points in a Q-dimensional objective space and we do not have the canonical order of $\mathbb{R}$ anymore. The reader is referred to Ref. 13 as a general reference in multicriteria optimization. Recall that a point $x \in \mathbb{R}^2$ is called a Pareto location or Pareto optimal if there exists no $y \in \mathbb{R}^2$ such that $F^q(y) \leq F^q(x)$, $\forall q \in \mathcal{Q}$, and $F^p(y) < F^p(x)$, for some $p \in \mathcal{Q}$, where $\mathcal{Q} := \{1, \dots, Q\}$. We denote the set of Pareto solutions by $\mathcal{X}^*_{\mathrm{Par}}(F^1, \dots, F^Q)$ or simply by $\mathcal{X}^*_{\mathrm{Par}}$ if this is possible without causing confusion.

For technical reasons, we will use also the concepts of weak Pareto optimality and strict Pareto optimality. A point $x \in \mathbb{R}^2$ is called a weak Pareto location or weakly Pareto optimal if there exists no $y \in \mathbb{R}^2$ such that $F^q(y) < F^q(x)$, $\forall q \in \mathcal{Q}$. We denote the set of weak Pareto solutions by $\mathcal{X}^*_{\mathrm{w-Par}}(F^1, \dots, F^Q)$ or simply by $\mathcal{X}^*_{\mathrm{w-Par}}$ if this is possible without causing confusion. A point $x \in \mathbb{R}^2$ is called a strict Pareto location or strictly Pareto optimal if there exists no $y \in \mathbb{R}^2$ such that $F^q(y) \leq F^q(x)$, $\forall q \in \mathcal{Q}$. Analogously, the set of strict Patero solutions is denoted by $\mathcal{X}^*_{\mathrm{s-Par}}(F^1, \dots, F^Q)$ or simply by $\mathcal{X}^*_{\mathrm{s-Par}}$ if this is possible without causing confusion. Note that

$$\mathcal{X}^*_{\mathrm{s-Par}} \subseteq \mathcal{X}^*_{\mathrm{Par}} \subseteq \mathcal{X}^*_{\mathrm{w-Par}}.$$

In our proofs, we use the concept of level sets. For a function $F: \mathbb{R}^2 \to \mathbb{R}$, the level set for a value $\rho \in \mathbb{R}$ is given by

$$L_{\leq}(F, \rho) := \left\{ x \in \mathbb{R}^2 : F(x) \leq \rho \right\}$$

and the level curve for a value $\rho \in \mathbb{R}$ is given by

$$L_{=}(F, \rho) := \left\{ x \in \mathbb{R}^2 : F(x) = \rho \right\}.$$

Using the level sets and level curves, Ref. 2 obtained the following characterizations:

$$x \in \mathcal{X}^*_{\mathrm{w-Par}}(F^1, \dots, F^Q) \Leftrightarrow \bigcap_{q=1}^{Q} L_{<}(F^q, F^q(x)) = \emptyset, \tag{3}$$

$$x \in \mathcal{X}^*_{\mathrm{Par}}(F^1, \dots, F^Q) \Leftrightarrow \bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(x)), \tag{4}$$

$$x \in \mathcal{X}^*_{\mathrm{s-Par}}(F^1, \dots, F^Q) \Leftrightarrow \bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \{x\}. \tag{5}$$

Finally, we recall that Ref. 14, proved the connectedness of the set $\mathcal{X}^*_{\mathrm{Par}}$.

3. Bicriteria Ordered Median Problems

In this section, we restrict ourselves to the bicriteria case, which is the basis for solving the Q-criteria case. To this end, we are looking for the Pareto solutions of the following vector optimization problem in $\mathbb{R}^2$:

$$\min_{x\in\mathbb{R}^2}\left[F^1(x):=\sum_{i=1}^{M}\lambda_i^1 d_{(i)}^1(x),\ F^2(x):=\sum_{i=1}^{M}\lambda_i^2 d_{(i)}^2(x)\right],$$

where the weights λ_i^q are in increasing order with respect to the index i for each $q=1,2$; that is,

$$\lambda_1^q\le\lambda_2^q\le\cdots\le\lambda_M^q,\quad q=1,2,$$

and $d_{(i)}^q(x)$ depends on the set Ω^q and the importance given to the existing facilities by the qth criterion, $q=1,2$. Therefore, the previous vector optimization problem is convex, see Section 2.

Note that, in a multicriteria setting, each objective function F^q, $q\in\mathcal{Q}$, generates its own set of bisector lines. Therefore, in the multicriteria case, the ordered elementary convex sets are generated by all the fundamental directions $d_g^i, i=1,\dots,M, g=1,\dots,G_i$, and the bisector lines $B^q(a_i,a_j)$, $q\in\mathcal{Q}$.

The following theorem characterizes geometrically the set $\mathcal{X}^*_{\text{Par}}$.

Theorem 3.1. $\mathcal{X}^*_{\text{Par}}(F^1,F^2)$ is a connected chain from $\mathcal{X}^*(F^1)$ to $\mathcal{X}^*(F^2)$ consisting of facets or vertices of cells or complete cells.

Proof. First of all,

$$\mathcal{X}^*(F^q)\neq\emptyset,\quad \text{for } q=1,2;$$

see Ref. 8. Moreover,

$$\mathcal{X}^*_{\text{Par}}\cap\mathcal{X}^*(F^q)\neq\emptyset,\quad \text{for } q=1,2;$$

therefore, we have that $\mathcal{X}^*_{\text{Par}}\neq\emptyset$, so we can choose $x\in\mathcal{X}^*_{\text{Par}}$. There exists at least one cell $\mathrm{C}\in\mathcal{C}$ with $x\in\mathrm{C}$. Recall that the functions F^1 and F^2 are linear within each cell (see Ref. 9). Hence, three cases occur.

(C1) $x\in\text{int (C)}$. Since $x\in\mathcal{X}^*_{\text{Par}}$, we obtain

$$\bigcap_{q=1}^{Q}L_{\le}(F^q,F^q(x))=\bigcap_{q=1}^{Q}L_{=}(F^q,F^q(x));$$

by the linearity of the ordered median problem in each cell, we have

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(y)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(y)), \quad \forall y \in \mathrm{C},$$

which means that $y \in \mathcal{X}^*_{\text{Par}}$, $\forall y \in \mathrm{C}$; hence, $\mathrm{C} \subseteq \mathcal{X}^*_{\text{Par}}$.

(C2) $x \in \overline{ab} := \mathrm{conv}\{a, b\} \subset \mathrm{bd(C)}$ and $a, b \in \mathrm{Ext(C)}$. We can choose $y \in \mathrm{int(C)}$ and two cases can occur:

(a) $y \in \mathcal{X}^*_{\text{Par}}$. Hence, we can continue as in Case 1.
(b) $y \notin \mathcal{X}^*_{\text{Par}}$. Using the linearity, we obtain first

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(z)) \neq \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(z)), \quad \forall z \in \mathrm{int(C)};$$

second, since $x \in \mathcal{X}^*_{\text{Par}}$, we have

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(z)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(z)), \quad \forall z \in \overline{ab}.$$

Therefore, we have that $\mathrm{C} \nsubseteq \mathcal{X}^*_{\text{Par}}$ and $\overline{ab} \subseteq \mathcal{X}^*_{\text{Par}}$.

(C3) $x \in \mathrm{Ext(C)}$. We can choose $y \in \mathrm{int(C)}$ and two cases can occur:

(a) If $y \in \mathcal{X}^*_{\text{Par}}$, we can continue as in Case 1.
(b) If $y \notin \mathcal{X}^*_{\text{Par}}$, we choose $z_1, z_2 \in \mathrm{Ext(C)}$ such that $\overline{xz_1}$, $\overline{xz_2}$ are faces of C.

(i) If z_1 or z_2 are in $\mathcal{X}^*_{\text{Par}}$, we can continue as in Case 2.
(ii) If z_1 and z_2 are not in $\mathcal{X}^*_{\text{Par}}$, then using the linearity in the same way as before, we obtain that $(\mathrm{C} \backslash \{x\}) \cap \mathcal{X}^*_{\text{Par}} = \emptyset$.

Hence, we obtain that the set of Pareto solutions consists of complete cells, complete faces, and vertices of these cells. Since we know that the set $\mathcal{X}^*_{\text{Par}}$ is connected, the proof is completed. □

In the following, we develop an algorithm to solve the bicriteria ordered median problem. The idea of this algorithm is to start in a vertex x of the cell structure which belongs to $\mathcal{X}^*_{\text{Par}}$, say

$$x \in \mathcal{X}^*_{1,2} := \arg \min_{x \in \mathcal{X}^*(F_1)} F_2(x),$$

the set of optimal lexicographical locations; see Ref. 15. Then, using the connectivity of $\mathcal{X}^*_{\mathrm{Par}}$, the algorithm proceeds by moving from vertex x to another Pareto optimal vertex y of the cell structure which is connected with the previous one by an elementary convex set. This procedure is repeated until the end of the chain reaches

$$\mathcal{X}^*_{2,1} := \arg\min_{x\in\mathcal{X}^*(F_2)} F_1(x).$$

Let C be a cell and let y, x, z be three vertices of C enumerated counterclockwise. By the linearity of the level sets, in each cell we can distinguish the following disjoint cases, if $x \in \mathcal{X}^*_{\mathrm{Par}}$:

(A) $C \subseteq \mathcal{X}^*_{\mathrm{Par}}$, i.e. C is contained in the chain.
(B) $\overline{xy}$ and $\overline{xz}$ are candidates for $\mathcal{X}^*_{\mathrm{Par}}$ and $\mathrm{int}(C) \nsubseteq \mathcal{X}^*_{\mathrm{Par}}$.
(C) $\overline{xy}$ is candidate for $\mathcal{X}^*_{\mathrm{Par}}$ and $\overline{xz}$ is not contained in $\mathcal{X}^*_{\mathrm{Par}}$.
(D) $\overline{xz}$ is candidate for $\mathcal{X}^*_{\mathrm{Par}}$ and $\overline{xy}$ is not contained in $\mathcal{X}^*_{\mathrm{Par}}$.
(E) Neither $\overline{xy}$ nor $\overline{xz}$ are contained in $\mathcal{X}^*_{\mathrm{Par}}$.

We denote by sit(C, x) the case (**A, B, C, D, E**) in which the cell C is classified with respect to the classification above, according to the extreme point x of C. The following lemma, whose proof is based on an exhaustive case analysis of the different relative positions of x within C, can be found in Ref. 16. It states when a given segment belongs to the Pareto set in terms of the sit$(\cdot,\cdot)$ function.

Lemma 3.1. Let $C_1 \ldots, C_{P_x}$ be the cells containing the intersection point x, considered in counterclockwise order, and let $y_1, \ldots, y_{P_x}$ be the intersection points adjacent to x, considered in counterclockwise order. If $x \in \mathcal{X}^*_{\mathrm{Par}}$ and $i \in \{1, \ldots, P_x\}$, then the following holds:

$$\overline{xy_{i+1}} \subseteq \mathcal{X}^*_{\mathrm{Par}} \Longleftrightarrow \begin{cases} & \mathrm{sit}(C_i, x) = \mathbf{A}, \\ \text{or} & \mathrm{sit}(C_{i+1}, x) = \mathbf{A}, \\ \text{or} & \begin{cases} \mathrm{sit}(C_i, x) \in \{\mathbf{B}, \mathbf{C}\}, \\ \mathrm{sit}(C_{i+1}, x) \in \{\mathbf{B}, \mathbf{D}\}. \end{cases} \end{cases}$$

These results allow us to describe the following algorithm.

Algorithm 3.1.

Step 1. Compute the planar graph generated by the cells and the two sets of lexicographical locations $\mathcal{X}^*_{1,2}$ and $\mathcal{X}^*_{2,1}$.

Step 2. If $\mathcal{X}^*_{1,2} \cap \mathcal{X}^*_{2,1} \neq \emptyset$, then set $\mathcal{X}^*_{\mathrm{Par}} := \mathrm{co}\{\mathcal{X}^*_{1,2}\}$ [trivial case $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) \neq \emptyset$]. Otherwise, set $\mathcal{X}^*_{\mathrm{Par}} := \mathcal{X}^*_{1,2} \cup \mathcal{X}^*_{2,1}$ [nontrivial case $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) = \emptyset$].

Step 3. Choose $x \in \mathcal{X}^*_{1,2} \cap \mathcal{GIP}$.

Step 4. While $x \notin \mathcal{X}^*_{2,1}$, repeat the following loop: scan the list of cells adjacent to x until we get situation **A** for a cell C or two consecutive cells C and $\overline{\mathrm{C}}$ in situations $\mathrm{C} \in \{\mathbf{B}, \mathbf{C}\}$ and $\overline{\mathrm{C}} \in \{\mathbf{B}, \mathbf{D}\}$, respectively.

Step 5. If situation **A** occurs, then $\mathcal{X}^*_{\mathrm{Par}} := \mathcal{X}^*_{\mathrm{Par}} \cup \mathrm{C}$; we have found a bounded cell. Otherwise, $\mathcal{X}^*_{\mathrm{Par}} := \mathcal{X}^*_{\mathrm{Par}} \cup \overline{xy}$; we have found a bounded face.

Step 6. Let C be the last scanned cell. Choose $y \in \mathcal{GIP} \cap C$ and such that y is connected to x. Set $x := y$ and go to Step 4.

Output: $\mathcal{X}^*_{\mathrm{Par}}(F^1, F^2)$. □

Reference 17 proved that the computation of a planar graph, induced by n lines in the plane, can be done in $O(n^2)$ time. This implies that, in the case of the ordered median problem, the computation of the planar graph generated by the fundamental directions and bisector lines is doable in $O(M^4 G^2_{\max})$ time.

The evaluation of the ordered median function needs $O(M \log(M G_{\max}))$ time for one point; therefore, we obtain $O(M^5 G^2_{\max} \log(M G_{\max}))$ time for the computation of lexicographic solutions. At the end, the complexity for computing the chain is $O(M^5 G^2_{\max} \log(M G_{\max}))$, since we have to consider at most $O(M^4 G^2_{\max})$ cells and the determination of sit(. , .) can be done in $O(M \log(M G_{\max}))$ time. The overall complexity is $O(M^5 G^2_{\max} \log(M G_{\max}))$ time. Notice that the polynomial complexity of this algorithm allows the efficient computation of the solution set.

Example 3.1. Consider a bicriteria problem with 20 existing facilities $A = \{a_1, \ldots, a_{20}\}$; see Figure 2. The coordinates $a_m = (x_m, y_m)$ of the existing facilities are given by the set

$$\{(1,7), (2,19), (7,14), (7,44), (8,6), (9,23), (10,33), (11,48), (14,1), (14,13), (16,36), (17,43), (19,9), (22,20), (24,34), (25,45), (27,4), (28,49), (29,28), (31,37)\}$$

and the weights $\omega^q_m, q = 1, 2$, are given by

$$w^1 = (10, 5, 5, 3, 15, 3, 1, 1, 10, 7, 1, 1, 5, 3, 0, 0, 7, 2, 2, 2),$$
$$w^2 = (3, 4, 1, 5, 1, 2, 6, 10, 0, 3, 5, 6, 2, 2, 5, 10, 2, 15, 10, 7).$$

All the existing facilities are associated with the 1-norm. Moreover, we assume that F^1 is a Weber function,

$$\lambda^1_m = 1, \quad \text{for all} \quad m = 1, \ldots, 20,$$

while F^2 is a center function,

$$\lambda_m^2 = 0, \quad \text{for } m = 1, \ldots, 19, \quad \lambda_{20}^2 = 1.$$

The optimal solution of the Weber function F^1 is unique and given by $\mathcal{X}_1^* = \{(10, 7)\}$ with the (optimal) objective value $z_1^* = 1344$. Therefore, we have $\mathcal{X}_{1,2}^* = \mathcal{X}_1^*$.

However, the optimal solution of the center function F^2 is given by the segment

$$\mathcal{X}_2^* = \overline{\left(23\tfrac{1}{6}, 41\tfrac{1}{6}\right)\left(25\tfrac{1}{4}, 43\tfrac{1}{4}\right)},$$

with (optimal) objective value $z_2^* = 190$. The center solution $\mathcal{X}_2^*$ lies on the bisector generated by (a_8, ω_8^2) and (a_{19}, ω_{19}^2). Moreover, we have

$$\mathcal{X}_{2,1}^* = \left\{\left(23\tfrac{1}{6}, 41\tfrac{1}{6}\right)\right\}.$$

The bicriteria chain, consisting of 5 cells and 14 edges with respect to the fundamental directions and bisectors drawn in Figure 2, is

$$\begin{aligned}\mathcal{X}_{\mathrm{Par}}^* = {} & \mathrm{conv}\{(10, 7), (11, 7), (11, 9), (10, 9)\} \cup \overline{(11, 9)(14, 9)} \cup \overline{(14, 9)(14, 19)} \\ & \cup \overline{(14, 19)(19, 19)} \cup \overline{(19, 19)(19, 20)} \cup \mathrm{conv}\{(19, 20), (22, 20), (22, 23), (19, 23)\} \\ & \cup \mathrm{conv}\{(22, 23), (26\tfrac{4}{5}, 23), (25\tfrac{4}{5}, 28)\} \cup \overline{(25\tfrac{4}{5}, 28)(23\tfrac{1}{6}, 41\tfrac{1}{6})}.\end{aligned}$$

Notice that the face $\overline{(26\tfrac{4}{5}, 23), (25\tfrac{4}{5}, 28)}$ and the segment $\overline{(25\tfrac{4}{5}, 28)(23\tfrac{1}{6}, 41\tfrac{1}{6})}$ are part of the bisector generated by the existing facilities $a_8 = (11, 48)$ and $a_{18} = (28, 49)$ with respect to $\omega_8^2 = 10$ and $\omega_{18}^2 = 15$.

Figure 2 shows the existing facilities, the fundamental directions, the bicriteria chain, two [out of $2\binom{M}{2} = 380$] bisectors, and four inflated unit balls determining the center solution.

4. Three-Criteria Case

In this section, we turn to the 3-criteria case and develop an efficient algorithm for computing $\mathcal{X}_{\mathrm{Par}}^*(F^1, F^2, F^3)$ using the results of the bicriteria case. We obtain a characterization of the Pareto solution set for the three-criteria case using the region surrounded by the chains of bicriteria Pareto solutions.

We denote

$$C_\infty(\mathbb{R}_0^+, \mathbb{R}^2) := \left\{\varphi | \varphi : \mathbb{R}_0^+ \to \mathbb{R}^2, \varphi \text{ continuous}, \lim_{t\to\infty} l_2(\varphi(t)) = +\infty\right\}, \quad (6)$$

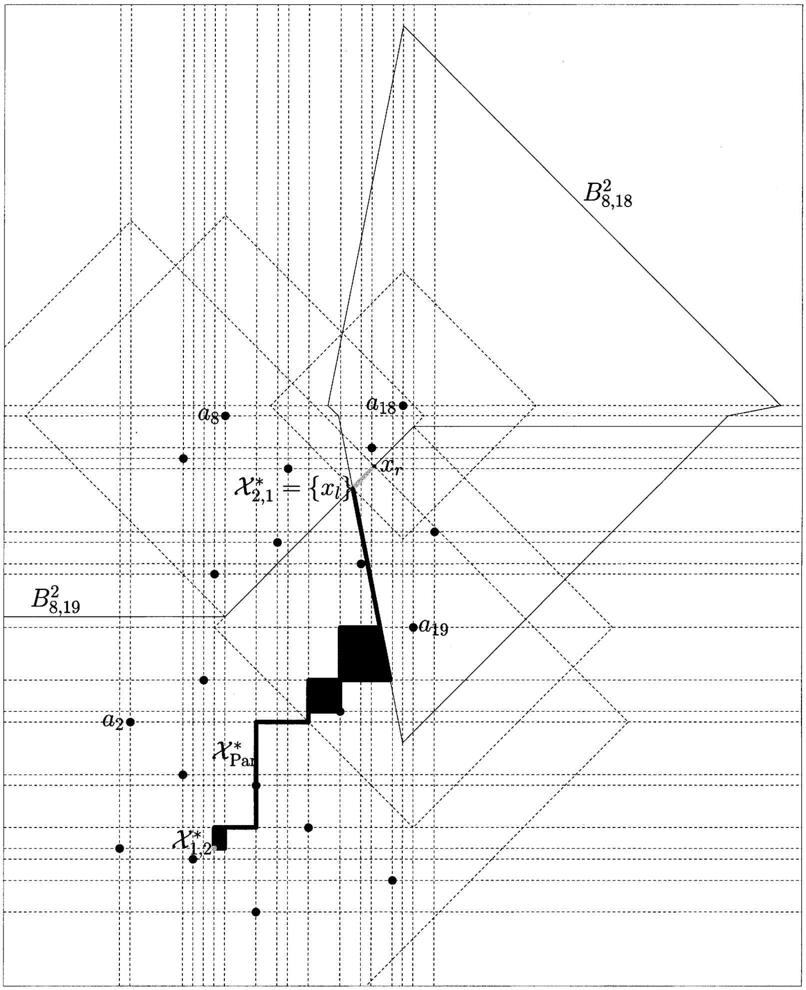

Fig. 2. Illustration to Example 3.1.

where $l_2(x)$ is the Euclidean norm of the point x and $C_\infty(\mathbb{R}_0^+, \mathbb{R}^2)$ is the set of continuous curves, which maps the set of nonnegative numbers $\mathbb{R}_0^+ := [0, \infty)$ into the two-dimensional space $\mathbb{R}^2$ and whose image $\varphi(\mathbb{R}_0^+)$ is unbounded in $\mathbb{R}^2$. These curves are introduced to characterize the geometrical locus of the points surrounded by Pareto chains.

For a set $S \subseteq \mathbb{R}^2$, we define the enclosure of S by

$$\operatorname{encl}(S) := \Big\{ x \in \mathbb{R}^2 : \exists \varepsilon > 0 \text{ with } B(x, \varepsilon) \cap S = \emptyset, \exists t_\varphi \in [0, \infty) \text{ with } \varphi\,(t_\varphi) \in S \text{ for all } \varphi \in C_\infty(\mathbb{R}_0^+, \mathbb{R}^2) \text{ with } \varphi(0) = x \Big\}, \tag{7}$$

where

$$B(x, \varepsilon) = \Big\{ y \in \mathbb{R}^2 : l_2(y - x) \leq \varepsilon \Big\}.$$

Notice that

$$S \cap \operatorname{encl}(S) = \emptyset.$$

Informally spoken, encl(S) contains all the points which are surrounded by S, but do not belong to S.

Lemma 4.1. If $x \in \mathbb{R}^2$ is dominated by $y \in \mathbb{R}^2$ with respect to strict Pareto optimally, then $z_\lambda := x + \lambda(x - y) \in \mathbb{R}^2$, with $\lambda \geq 0$, is dominated by x with respect to strict Pareto optimality.

Proof. From

$$z_\lambda := x + \lambda(x - y) \quad \text{and} \quad \lambda \geq 0,$$

it follows that

$$x = \frac{1}{1+\lambda} z_\lambda + \frac{\lambda}{1+\lambda} y,$$

with $\lambda \geq 0$. Since y dominates x with respect to strict Pareto optimality and because of the convexity of $F^1, \dots, F^Q$, we obtain

$$\begin{aligned} F^q(x) &\leq [1/(1+\lambda)]\, F^q(z_\lambda) + [\lambda/(1+\lambda)]\, F^q(y) \\ &< [1/(1+\lambda)]\, F^q(z_\lambda) + [\lambda/(1+\lambda)]\, F^q(x), \quad \text{for all } q \in \mathcal{Q}, \end{aligned}$$

which implies that

$$(1+\lambda) F^q(x) < F^q(z_\lambda) + \lambda F^q(x), \quad \text{for all } q \in \mathcal{Q}.$$

Hence,

$$F^q(x) < F^q(z_\lambda), \quad \text{for all } q \in \mathcal{Q};$$

i.e., x dominates z_λ with respect to strict Pareto optimality. □

We denote the union of the bicriteria chains including the 1-criterion solutions by

$$\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3) := \bigcup_{q=1}^{3} \mathcal{X}^*(F^q) \cup \bigcup_{q=1}^{2} \bigcup_{p=q+1}^{3} \mathcal{X}_{\text{Par}}^*(F^p, F^q). \tag{8}$$

We use gen, since this set will generate the set $\mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3)$; see Figure 3.

The next lemma provides usefull geometric information to build $\mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3)$.

Also, we learn more about the part of the plane which is crossed by the Pareto chains. For a set A, let $\text{cl}(A)$ denote the topological closure of A.

Lemma 4.2. The following inclusion of sets holds:

$$\text{cl}\left(\text{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}\left(F^1, F^2, F^3\right)\right)\right) \subseteq \mathcal{X}_{\text{s-Par}}^*\left(F^1, F^2, F^3\right).$$

Proof. Let

$$x \in \text{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}\left(F^1, F^2, F^3\right)\right).$$

Assume that

$$x \notin \mathcal{X}_{\text{s-Par}}^*\left(F^1, F^2, F^3\right).$$

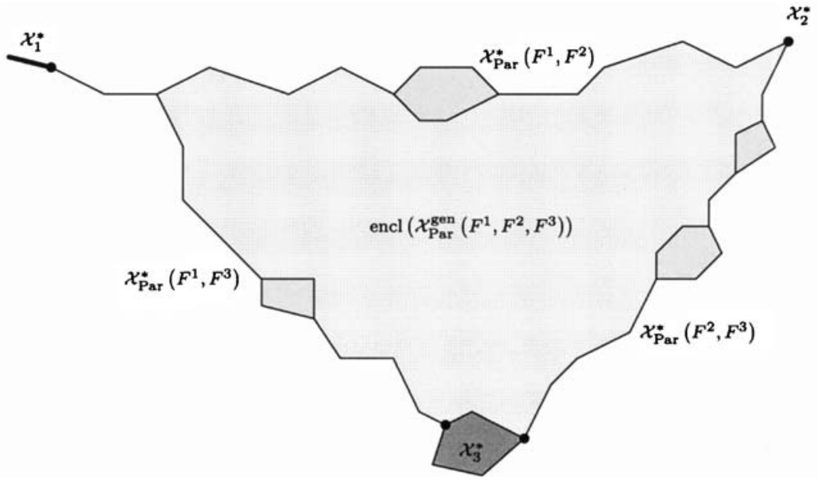

Fig. 3. Enclosure of $\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3)$.

Then, there exists a point $y\in\mathbb{R}^2$ which dominates x with respect to strict Pareto optimality. Consider the curve $\varphi:\mathbb{R}_0^+\to\mathbb{R}^2$ such that

$$\varphi(t)=x+t(x-y).$$

Obviously, φ is continuous and fulfills

$$\lim_{t\to\infty} l_2((\varphi(t))=+\infty, \quad \text{i.e., } \varphi\in C_\infty(\mathbb{R}_0^+,\mathbb{R}^2).$$

Moreover $\varphi(0)=x$. Since

$$x\in \operatorname{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}\left(F^1,F^2,F^3\right)\right),$$

there exists $t\in[0,\infty)$ with

$$z_t:=\varphi(t)\in\mathcal{X}_{\text{Par}}^{\text{gen}}\left(F^1,F^2,F^3\right). \tag{9}$$

By Lemma 4.1, we have

$$F^q(x)\le F^q(z_t), \quad \text{for all } q\in\mathcal{Q}.$$

Hence, we can continue with the following case analysis with respect to (9):

Case 1. $z_t\in\mathcal{X}^*(F^q)$, for some $q\in\{1,2,3\}$: $\Rightarrow x\in\mathcal{X}^*(F^q)\ \Rightarrow x\in\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1,F^2,F^3)$. This is a contradiction.

Case 2. $z_t\in\mathcal{X}_{\text{Par}}^*(F^p,F^q)$, for some $p,q\in\{1,2,3\}$, $p<q$:
$\Rightarrow x\in\mathcal{X}_{\text{Par}}^*(F^p,F^q)\Rightarrow x\in\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1,F^2,F^3)$.
This is a contradiction.

Therefore,

$$x\in\mathcal{X}_{\text{s}-\text{Par}}^*\left(F^1,F^2,F^3\right);$$

i.e.,

$$\operatorname{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}\left(F^1,F^2,F^3\right)\right)\subseteq\mathcal{X}_{\text{s}-\text{Par}}^*\left(F^1,F^2,F^3\right).$$

Since $\mathcal{X}_{\text{s}-\text{Par}}^*(F^1,F^2,F^3)$ is closed (see Ref. 18, Chapter 4, Theorem 27), we obtain

$$\operatorname{cl}\Big(\operatorname{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}\left(F^1,F^2,F^3\right)\right)\Big)\subseteq\operatorname{cl}\left(\mathcal{X}_{\text{s}-\text{Par}}^*\left(F^1,F^2,F^3\right)\right)=\mathcal{X}_{\text{s}-\text{Par}}^*\left(F^1,F^2,F^3\right).$$

$\square$

Finally, we obtain the following theorem which provides a subset as well as a superset of $\mathcal{X}^*_{\text{Par}}(F^1, F^2, F^3)$.

Theorem 4.1. The following inclusions of sets holds:

$$\begin{aligned}\operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)\right) &\subseteq \mathcal{X}^*_{\text{Par}}\left(F^1, F^2, F^3\right)\\ &\subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right) \cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)\right).\end{aligned}$$

Proof. Using Lemma 4.2 and Theorem 3.2 in Ref. 5, we have the following chain of inclusions that proves the thesis of the theorem:

$$\begin{aligned}\operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)\right) &\subseteq \mathcal{X}^*_{\text{s-Par}}\left(F^1, F^2, F^3\right)\\ &\subseteq \mathcal{X}^*_{\text{Par}}\left(F^1, F^2, F^3\right) \subseteq \mathcal{X}^*_{\text{w-Par}}\left(F^1, F^2, F^3\right)\\ &\subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right) \cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)\right). \qquad \square\end{aligned}$$

Now, it remains to consider the Pareto optimality of the set $\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3)$ with respect to the three objective functions F^1, F^2, F^3. For a cell $C \in \mathcal{C}$, we define the collapsing and the remaining part of C with respect to Q-criteria optimality by

$$\operatorname{col}_Q(C) := \{x \in C : x \notin \mathcal{X}^*_{\text{par}}(F^1, \ldots, F^Q)\}, \tag{10}$$

$$\operatorname{rem}_Q(C) := \{x \in C : x \in \mathcal{X}^*_{\text{par}}(F^1, \ldots, F^Q)\}. \tag{11}$$

Using the differentiability of the objective functions in the interior of the cells, we obtain the following lemma.

Lemma 4.3. Any $C \in \mathcal{C}$ satisfies:

(i) $\operatorname{col}_Q(C) \,\dot{\cup}\, \operatorname{rem}_Q(C) = C$.

(ii) Either $\operatorname{rem}_Q(C) = C$ or $\operatorname{rem}_Q(C) \subseteq \operatorname{bd}(C)$. In the latter case $\operatorname{rem}_Q(C)$ is either empty or consists of complete faces and/or extreme points of C.

(iii) For $C \subseteq \mathcal{X}^*(F^p)$, with $p \in \{1, 2, 3\}$ and $x \in \operatorname{int}(C)$, we have

$$x \in \operatorname{rem}_3(C) \Leftrightarrow \left\{\begin{array}{l}\exists\, \xi \in \mathbb{R}, \text{ with } \nabla F^q(x) = \xi \nabla F^r(x)\\ \text{and } \ \xi < 0, \ \text{ for } \ q, r \in \{1, 2, 3\} \backslash \{p\}, q < r\end{array}\right\}. \tag{12}$$

For $C \subseteq \mathcal{X}^*_{\mathrm{Par}}(F^p, F^q)$, with $p, q \in \{1,2,3\}, p<q$, and $x \in \mathrm{int}(C)$, we have

$$x \in \mathrm{rem}_3(C)$$
$$\Leftrightarrow \left\{ \begin{array}{l} \exists \xi^p, \xi^q \in \mathbb{R} \text{ with } \nabla F^r(x) = \xi^p \nabla F^p(x), \\ \nabla F^r(x) = \xi^q \nabla F^q(x) \text{ and } \xi^p \xi^q \leq 0, \text{ for } r \in \{1,2,3\} \backslash \{p,q\} \end{array} \right\}. \tag{13}$$

Proof. This follows directly from the definition of $\mathrm{col}_Q(C)$ and $\mathrm{rem}_Q(C)$.

(i) If $\mathrm{int}(C) \cap \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3) \neq \emptyset$, we have $C \subseteq \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3)$ and hence $\mathrm{rem}_Q(C) = C$. This follows analogously to the proof of Theorem 3.1.

(ii) If $\mathrm{int}(C) \cap \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3) = \emptyset$, then $\mathrm{rem}_Q(C) \subseteq \mathrm{bd}(C)$. The rest follows analogously to the proof of Theorem 3.1.

(iii) If $C \subseteq \mathcal{X}^*(F^p)$, for $p \in \{1,2,3\}$ and $x \in \mathrm{int}(C)$, we have $L_=(F^p, F^p(x)) = L_\leq(F^p, F^p(x))$; therefore, there exist $q, r \in \{1,2,3\} \backslash \{p\}, q<r$, such that

$$\begin{aligned} x \in \mathrm{rem}_3(C) \Leftrightarrow & L_=(F^p, F^q(x)) \cap L_=(F^r, F^r(x)) \\ & = L_\leq(F^q, F^q(x)) \cap L_\leq(F^r, F^r(x)) \\ & \Leftrightarrow \nabla F^q(x) = \xi \nabla F^r(x), \text{ with } \xi < 0. \end{aligned}$$

If $C \subseteq \mathcal{X}^*_{\mathrm{Par}}(F^p, F^q)$, for $p, q \in \{1,2,3\}$ and $x \in \mathrm{int}(C)$, there exists $\xi \in \mathbb{R}$ with $\nabla F^p(x) = \xi \nabla F^p(x)$ with $\xi < 0$. Notice that the trivial case $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) \neq \emptyset$, i.e., $\nabla F^q(x) = 0 = \nabla F^q(x)$, is included.

The Pareto optimality condition

$$\bigcap_{q=1}^{3} L_=(F^q, F^q(x)) = \bigcap_{q=1}^{3} L_\leq(F^q, F^q(x))$$

for the 3 criteria is fulfilled if and only if the level curve $L^r_=(x)$, $r \in \{1,2,3\} \backslash \{p,q\}$ has the same slope as $L_=(F^p, F^p(x))$ and $L_=(F^q, F^q(x))$; i.e., if and only if $\xi^p, \xi^q \in \mathbb{R}$ exist satisfying

$$\nabla F^r(x) = \xi^p \nabla F^p(x), \quad \nabla F^r(x) = \xi^q \nabla F^q(x), \quad \xi^p \xi^q \leq 0. \qquad \square$$

Summing up the preceding results, we get a complete geometric characterization of the set of Pareto solutions for the three criteria case.

Theorem 4.2. The set of Pareto solutions satisfies

$$\mathcal{X}^*_{\mathrm{Par}}\left(F^1, F^2, F^3\right) = \left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right) \cup \mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right)\right)\right) \setminus \left\{x \in \mathbb{R}^2 : \exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right), x \in \mathrm{col}_3(C)\right\}.$$

Proof. Let

$$y \in \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3).$$

Then, by Theorem 4.1, we have that

$$y \in \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right) \cup \mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right)\right).$$

Moreover, for $C \in \mathcal{C}$, with $y \in C$, we have

$$y \in \mathrm{rem}_3(C), \quad \text{i.e., } y \notin \mathrm{col}_3(C).$$

This implies

$$y \in \left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right) \cup \mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right)\right)\right) \setminus \left\{x \in \mathbb{R}^2 : \exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right), x \in \mathrm{col}_3\,(C)\right\}.$$

Now, let y belonging to the set above; we distinguish the following cases:

Case 1. $y \in \mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3))$. Then, $y \in \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3)$ by Theorem 4.1.

Case 2. $y \in \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3)$.

Case 2.1. $\exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3)$, with $y \in C$,
$\Rightarrow y \notin \mathrm{col}_3(C) \Rightarrow y \in \mathrm{rem}_3(C) \Rightarrow y \in \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3)$.

Case 2.2. $\nexists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3)$, with $y \in C$,
$\Rightarrow L_{\le}(F^p, F^p(y)) \cap L_{\le}(F^q, F^q(y)) = \{y\}$, for some $p, q \in \{1, 2, 3\}$, $p < q$
$\Rightarrow \bigcap_{q=1}^{3} L_{\le}(F^q, F^q(y)) = \{y\} \Rightarrow y \in \mathcal{X}^*_{\mathrm{s-Par}}(F^1, F^2, F^3) \subseteq \mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3)$. □

In the case of ordered median functions, the gradients $\nabla F^q(x), q \in \{1, 2, 3\}$, in those points where they are well-defined, can be computed in $O(M \log(MG_{\max}))$ time (analogous to the evaluation of the function). Therefore, we can test with (12) and (13) in $O(M \log(MG_{\max}))$ time if a cell $C \in \mathcal{C}, C \subseteq \mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3)$, collapses. We obtain the following algorithm for the 3-criteria ordered median problem with time complexity $O(M^5 G_{\max}^2 \log(M G_{\max}))$.

Algorithm 4.1.

Step 1. Compute the planar graph generated by the cells $\mathcal{C}$ and $\mathcal{X}_{\text{w-Par}}^*(F^1, F^3), \mathcal{X}_{\text{w-Par}}^*(F^2, F^3)$ using Algorithm 3.1.

Step 2. Set $\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3) := \mathcal{X}_{\text{w-Par}}^*(F^1, F^2) \cup \mathcal{X}_{\text{w-Par}}^*(F^1, F^3) \cup \mathcal{X}_{\text{w-Par}}^*(F^2, F^3)$ and $\mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3) := \mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3) \cup \text{encl}(\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3))$.

Step 3. For any $C \in \mathcal{C}$, with $C \subseteq \mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3)$, compute $\text{col}_3(C)$ using Lemma 4.3 and set $\mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3) := \mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3) \backslash \text{col}_3(C)$.

Output: $\mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3)$.

Finally we present an example to illustrate the preceding results.

Example 4.1. Consider four existing facilities

$$a_1 = (2, 6.5), \quad a_2 = (5, 9.5), \quad a_3 = (6.5, 2), \quad a_4 = (11, 9.5);$$

see Figure 1. a_1 and a_4 are associated with the l_1-norm, whereas a_2 and a_3 are associated with the l_∞-norm. Moreover, we are given three ordered median functions F^q defined by the following weights:

$$\begin{aligned}
&\omega_1^1 = \omega_2^1 = \omega_4^1 = 1, \quad \omega_3^1 = 0,\\
&\omega_1^2 = \omega_2^2 = \omega_3^2 = 1, \quad \omega_4^2 = 0,\\
&\omega_1^3 = \omega_2^3 = \omega_3^3 = \omega_4^3 = 1,\\
&\lambda_1^1 = \lambda_2^1 = \lambda_3^1 = \lambda_4^1 = \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = \lambda_4^3 = 1,\\
&\lambda_1^3 = \lambda_2^3 = \lambda_3^3 = 0.
\end{aligned}$$

We obtain the optimal solutions

$$\mathcal{X}^*(F^1) = \{a_2\}, \ \ \mathcal{X}^*(F^2) = \{a_1\}, \ \ \mathcal{X}^*(F^3) = \overline{(6.5, 8), (8, 6.5)}.$$

The sets $\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3)$ and $\mathcal{X}_{\text{Par}}^*(F^1, F^2, F^3)$ are drawn in Figure 4. This figure shows a part of the whole situation presented in Figure 1.

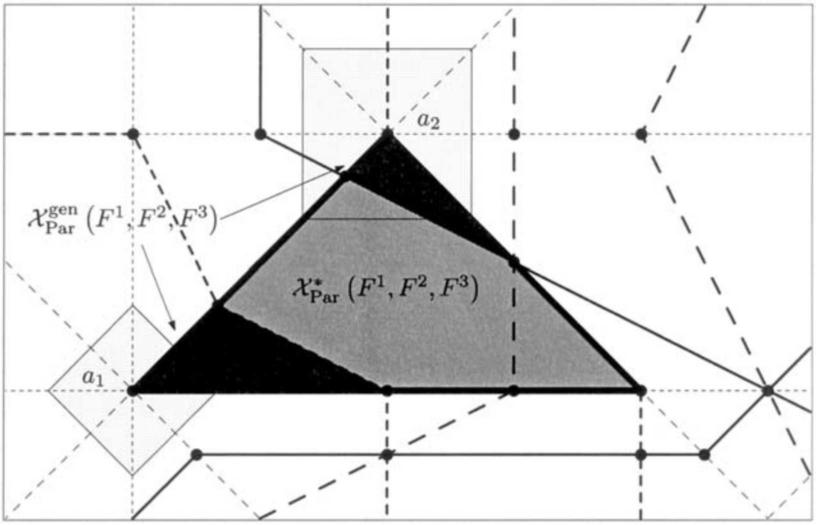

Fig. 4. Illustration to Example 4.1.

5. Case where $Q > 3$

In this section, we consider the general Q-criteria case $Q > 3$. We prove that the Pareto solution set can be obtained from the Pareto solution sets of the three criteria problem. This construction requires the removal of the dominated points from the union of all the three criteria Pareto solution sets. The reader may notice that all this construction reduces to obtaining the bicriteria Pareto chains as proved in Theorem 4.2.

Theorem 5.1. For the Q-criteria case, the following inclusions hold:

(i) $$\bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathrm{cl}(\mathrm{encl}(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^p, F^q, F^r))) \subseteq \mathcal{X}^{*}_{\text{Par}}(F^1, \ldots, F^Q).$$

(ii) $$\mathcal{X}^{*}_{\text{Par}}(F^1, \ldots, F^Q) \subseteq \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathcal{X}^{\text{gen}}_{\text{Par}}(F^p, F^q, F^r) \cup \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathrm{encl}(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^p, F^q, F^r)).$$

Proof.

(i) Let

$$x \in \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathrm{cl}(\mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r))).$$

This is equivalent to

$$x \in \mathrm{cl}(\mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r))), \quad \text{for some } p, q, r, \in \mathcal{Q}, \ p < q < r.$$

Then, by Lemma 4.2,

$$x \in \mathcal{X}^{*}_{\mathrm{s-Par}}(F^p, F^q, F^r), \quad \text{for some } p, q, r, \in \mathcal{Q}, \ p < q < r.$$

Applying (5), this is equivalent to

$$L_{\leq}(F^p, F^p(x)) \cap L_{\leq}(F^q, F^q(x)) \cap L_{\leq}(F^r, F^r(x)) = \{x\},$$
$$\text{for some } p, q, r, \in \mathcal{Q}, \ p < q < r;$$

since

$$x \in L_{\leq}(F^q, F^q(x)), \quad \text{for all } q \in \mathcal{Q},$$

it follows that

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \{x\}.$$

Finally, by (5),

$$x \in \mathcal{X}^{*}_{\mathrm{s-Par}}(F^1, \ldots, F^Q),$$

which implies that

$$x \in \mathcal{X}^{*}_{\mathrm{Par}}(F^1, \ldots, F^Q).$$

(ii) Let

$$x \in \mathcal{X}^{*}_{\mathrm{Par}}(F^1, \ldots, F^Q);$$

then,

$$x \in \mathcal{X}^{*}_{\mathrm{w-Par}}(F^1, \ldots, F^Q);$$

by (3), this is equivalent to

$$\bigcap_{q=1}^{Q} L_<(F^q, F^q(x)) = \emptyset.$$

By the Helly theorem, there exist $p, q, r \in \mathcal{Q}$, $p < q < r$, such that

$$L_<(F^p, F^p(x)) \cap L_<(F^q, F^q(x)) \cap L_<(F^r, F^r(x)) = \emptyset.$$

By (3), this is equivalent to

$$x \in \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q, F^r), \qquad \text{for some } p, q, r \in \mathcal{Q},\ p < q < r;$$

by Theorem 3.2 in Ref. 5, this implies that

$$x \in \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r) \cup \mathrm{encl}\,(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r)),$$
$$\text{for some } p, q, r \in \mathcal{Q},\ p < q < r.$$

Finally, this can be equivalently written as

$$x \in \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r) \cup \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathrm{encl}\ (\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r)). \qquad \square$$

In the Q-criteria case, the crucial region is now given by the cells $C \in \mathcal{C}$ with

$$\begin{aligned} C &\subseteq \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r) \quad \setminus \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r)) \\ &= \bigcup_{\substack{p,q \in \mathcal{Q} \\ p<q}} \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q) \quad \setminus \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p, F^q, F^r)). \end{aligned}$$

Similar to Lemma 4.3, by comparing the gradients of the objective functions in int(C), one can test whether or not the cell $C \in \mathcal{C}$ collapses with respect to $F^1, \ldots, F^Q$.

Finally, we obtain the following theorem, which can be proven by the same technique as in the 3-criteria case; see the proof of Theorem 4.2.

Theorem 5.2. For the Q-criteria case, the following result holds.

$$\mathcal{X}^*_{\mathrm{Par}}(F^1,\dots,F^Q)=\left(\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}}\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r)\cup\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}}\mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r))\right)\Bigg\backslash\left\{x\in\mathbb{R}^2:\exists C\in\mathcal{C},\, C\subseteq\bigcup_{\substack{p,q\in\mathcal{Q}\\ p<q}}\mathcal{X}^*_{\mathrm{w-Par}}(F^p,F^q)\Bigg\backslash\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}}\mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r)),\, x\in\mathrm{col}_Q(C)\right\}.$$

For the Q-criteria ordered median problem, we obtain the following algorithm.

Algorithm 5.1.

Step 1. Compute the planar subdivision generated by the cells $C\in\mathcal{C}$ and $\mathcal{X}^*_{\mathrm{w-Par}}(F^p,F^q)$, $p,q\in\mathcal{Q}$, $p<q$, using Algorithm 3.1.

Step 2. Set $\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r):=\mathcal{X}^*_{\mathrm{w-Par}}(F^p,F^q)\cup\mathcal{X}^*_{\mathrm{w-Par}}(F^p,F^r)\cup\mathcal{X}^*_{\mathrm{w-Par}}(F^q,F^r)$, $\forall p,q,r$, with $p<q<r$, $\mathcal{X}^*_{\mathrm{Par}}(F^1,\dots,F^Q):=\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}}\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r)\cup\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}}\mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r))$.

Step 3. For all cell $C\subseteq\bigcup_{\substack{p,q\in\mathcal{Q}\\ p<q}}\mathcal{X}^*_{\mathrm{w-Par}}(F^p,F^q)\backslash\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}}\mathrm{encl}(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^p,F^q,F^r))$, compute $\mathrm{col}_Q(C)$ and set $\mathcal{X}^*_{\mathrm{Par}}(F^1,\dots,F^Q):=\mathcal{X}^*_{\mathrm{Par}}(F^1,\dots,F^Q)\backslash\mathrm{col}_Q(C)$.

Output: $\mathcal{X}^*_{\mathrm{Par}}(F^1,\dots,F^Q)$.

The complexity of Algorithm 5.1 can be obtained by the following analysis. For each cell C, $\mathrm{col}_Q(C)$ takes $O(Q\log(Q))$. Algorithm 5.1 needs to solve $O(Q^3)$ three-criteria problems which dominate all other elementary operations of the algorithm. Each one of them has the same complexity as the two-criteria problem. Thus, the overall time complexity is

$$\begin{aligned}&O(M^5 G_{\max}^2 Q^3(\log M+\log G_{\max})+M^4 G_{\max}^2 Q\log Q)\\&=O(M^5 G_{\max}^2 Q^3(\log M+\log G_{\max}).\end{aligned}$$

6. Concluding Remarks

In this paper, we have shown the usefulness of the ordered median problems for modeling multicriteria locational decision problems. The characterization of the Pareto solution set of a general Q-criteria problem has been reduced to solving a series of bicriteria problems. Efficient algorithms have been developed together with a detailed complexity analysis.

Extensions of the results in the paper are worth to be investigated. If we allow the weights ω_i^q, $i\in\mathcal{M}$, $q\in\mathcal{Q}$, to be positive or negative and the weights λ_i^q, $i\in\mathcal{M}$, $q\in\mathcal{Q}$, not to be in nondecreasing sequence, we cannot apply the procedures presented in the preceding sections. Especially, we lose: (a) the convexity of the objective functions F^q, $q\in\mathcal{Q}$; (b) the connectivity of the set of Pareto optimal points $\mathcal{X}_{\mathrm{Par}}^*(F^1,\dots,F^q)$.

As a consequence, a solution algorithm for the multicriteria ordered median problem, with attraction and repulsion, has to have a completely different structure than the algorithm for the convex case. However, the following properties are still fulfilled: (c) the cell structure remains the same, since the fundamental directions and bisector lines do not depend on λ_i^q; (d) we still have the linearity of the objective functions F^q inside each cell.

Consequently, we can compute the local Pareto solutions with respect to a single cell as described in the previous sections. Of course, we cannot ensure that the local Pareto solutions remain globally Pareto. Therefore, to obtain the set of global Pareto solutions, all the local Pareto solutions have to be compared.

A schematic approach for solving the ordered median problem with positive and negative weights is to compute the local Pareto solutions for each $C\in\mathcal{C}$ and then to compare all solutions of Step 1 and get $\mathcal{X}_{\mathrm{Par}}^*(F^1,\dots,F^Q)$.

In general, comparing the local Pareto solutions might become very time consuming. However, for more special cases, efficient algorithms can be developed. If we restrict ourselves to the bicriteria case, we can do a procedure similar to the one used in Ref. 19 for network location problems. Further details can be found in Ref. 16.

Extensions to the multifacility case as well as improvements for the complexity results for special cases are under research. Also, a more detailed discussion of the problems mentioned in Section 6 is planned. Furthermore, we are working on an implementation of ordered median problems in LoLA (Library of Location Algorithms, Ref. 20).

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